Indian Statistical Institute
M.Math Second year
Back up exam - Partial differential Equations

Total Marks : 40

1. (a) Define the normed linear space $W_0^{k,p}(\Omega)$ for $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. [2]
   (b) Is it true that $C^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R} \setminus \{0\})$. [2]

2. (a) State Trace theorem for $H^1(\Omega)$. [2]
   (b) Let Ω be a bounded open set in $\mathbb{R}^N$ with C^1 boundary Γ. Let $u, v \in H^1(\Omega)$. Then for $1 \leq i \leq n$, show that
$$\int_\Omega u \frac{\partial v}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i} v + \int_\Gamma (\gamma_0 v)\,(\gamma_0 u)\,\nu_i dS$$
   where $\gamma_0 u$ and $\gamma_0 v$ denotes the trace of u and v respectively. [4]

3. Derive d'Alembert's formula for the wave equation in one space dimension. Deduce the smoothness of the solution in terms of the given data. [5]

4. Let $\Omega = (a, b)$ and $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of Ω. Let $I_k = (x_{k-1}, x_k)$ for $k = 1, 2 \cdots n$. Let $f : \Omega \to \mathbb{R}$ be such that $f|_{I_k} \in H^1(I_k)$ for each $0 \leq k \leq n-1$. Show that $f \in H^1(\Omega)$ if and only if $f \in C(\overline{\Omega})$. [5]

5. Let $1 < p \leq \infty$ and $u \in L^p(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. Show that $u \in W^{1,p}(\Omega)$ if there exists a constance C such that $|\int_\Omega u \frac{\partial \varphi}{\partial x_i}| \leq C\|\varphi\|_{L^{p'}(\Omega)}$ for every $\varphi \in C_c^\infty(\Omega)$ and $1 \leq i \leq n$, p' is the conjugate exponent of p. [5]

6. State and prove Lax-Milgram lemma. [5]

7. Fix an $\alpha > 0$ and $\Omega = B_1(0)$. Show that there exists a constant C depending only on n and α such that
$$\int_\Omega u^2 dx \leq C \int_\Omega |Du|^2 dx$$
   provided $|\{x \in \Omega : u(x) = 0\}| > \alpha$. [5]

8. Define hypoelliptic operator. Prove that $Lu = -\Delta u + u$ is a hypoelliptic operator in $\mathbb{R}^N$. [5]